

Giroux correspondence, confoliations, and symplectic structures on $S^1 \times M$

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Abstract

Let M be a closed oriented 3-manifold such that $S^1 \times M$ admits a symplectic structure ω . The goal of this paper is to show that M is a fiber bundle over S^1 . The basic idea is to use the obvious S^1 -action on $S^1 \times M$ by rotating the first factor of $S^1 \times M$, and one of the key steps is to show that the S^1 -action on $S^1 \times M$ is actually symplectic with respect to a symplectic form cohomologous to ω . We achieve it by crucially using the recent result or its relative version of Giroux about one-to-one correspondence between open book decompositions of M up to positive stabilization and co-oriented contact structures on M up to contact isotopy. As a consequence, we can give an answer to a question of Kronheimer concerning the relation between symplectic structures on $S^1 \times M_K$ and fibered knots K , where M_K denotes the result of 0-surgery on S^3 along a knot K in S^3 . Moreover, a complete picture of the various intriguing implications between symplectic structures on $S^1 \times M_K$ and fibered knots can be provided as in Table I below, and thus we fill in the missing links in the circle of ideas around this topic.

1 Introduction and statements of results

It was Thurston in [31] who first proved that any closed oriented smooth 4-manifold X which fibers over a Riemann surface admits a symplectic structure, unless the fiber class is torsion in $H_2(X, \mathbf{Z})$. Thus, if the genus of the fiber of such a closed oriented 4-manifold X is greater than or equal to 2, then the manifold X always admits a symplectic structure. Moreover, any fibration of a closed oriented 3-manifold M over a circle S^1 induces a symplectic structure on the 4-manifold $S^1 \times M$. (See [24] and [16].) Furthermore, it seems to have been widely believed that the converse also holds (see [30]).

Conjecture 1.1. *Let M be a closed oriented 3-manifold such that $S^1 \times M$ admits a symplectic structure. Then M fibers over S^1 .*

Indeed, there have been several attempts towards the conjecture, and it turns out that the conjecture is true for many important classes of 4-manifolds. For instance, see [22], [3], [5], [10], [11], and [12].

The goal of this paper is that by taking a completely different but elementary approach from the previous work, we give a short and affirmative proof to the Conjecture 1.1. To do so, we shall use the S^1 -action on the symplectic 4-manifold $S^1 \times M$ obtained by rotating the first factor of $S^1 \times M$. One crucial observation of the proof is that every symplectic class on $S^1 \times M$ can always be represented by a symplectic form invariant under the S^1 -action. We show this important fact by a recent result and its relative version of Giroux about one-to-one correspondence between contact structures up to contact isotopy and open book decompositions up to positive stabilizations (see [15], [18], [6], and [7] for more details).

To the author's knowledge, it is still unknown whether or not any action of a compact connected Lie group G on a symplectic $2n$ -manifold X always induces a G -invariant symplectic form on X , in general. On the other hand, in Riemannian geometry one can always obtain a G -invariant Riemannian metric by taking the average of a Riemannian metric over the Lie group G . As $S^2 \times S^2$ with the product symplectic form shows, simply taking the average of a symplectic form over the Lie group G does not yield a G -invariant symplectic form [27].

Once we show that there exists a symplectic form invariant under an S^1 -action on $S^1 \times M$, it is an easy and well-known procedure to complete the proof of the conjecture. In Section 4, for the sake of the reader we provide a proof of Theorem 4.2, using an argument of D. Tischler in [32] about fibering certain foliated manifolds over S^1 . We remark that Theorem 18 in [8] gives an alternative argument of the second half of the proof of our main Theorem 4.2.

As a generalization of the Theorem 4.2, Baldridge asked in [1] whether or not for every closed symplectic 4-manifold admitting a free S^1 -action whose orbit space is M the quotient manifold M fibers over S^1 . We think that our method of the present paper can be adapted to answer the following Conjecture 1.2. But we do not pursue it in this paper, for the sake of simplicity.

Conjecture 1.2. *Let X be a closed symplectic 4-manifold admitting a free S^1 -action whose orbit space is M . Then the quotient manifold M fibers over S^1 .*

Let us denote by M_K a 3-manifold M_K obtained by 0-surgery on a knot K in S^3 . As an interesting consequence, we can give an answer to the question in [20] of Kronheimer about symplectic structures on $S^1 \times M_K$. To be more precise, Fintushel and Stern proved in [9] that if $S^1 \times M_K$ admits a symplectic structure then the symmetrized Alexander polynomial $\Delta_K(t)$ is monic. On the other hand, Kronheimer proved in [20] that if the knot has a genus $g(K)$ of two or more a necessary condition for $S^1 \times M_K$ to admit a symplectic structure is that its genus $g(K)$ be equal to the degree of its symmetrized Alexander polynomial. It is a well-known fact ([2] or [28]) that if a knot K is fibered then its symmetrized Alexander polynomial is monic and its genus $g(K)$ is equal to the degree of its symmetrized Alexander polynomial. Since $S^1 \times M_K$ is symplectic for fibered knots, Kronheimer raised a question whether or not $S^1 \times M_K$ admits a symplectic structure for *non-fibered* knots such as the pretzel knot $P(5, -3, 5)$. The symmetrized Alexander polynomial of the pretzel knot $P(5, -3, 5)$ is $t - 3 + t^{-1}$ and thus monic with its degree equal to the genus 1 of the knot. Our another main result is to give a negative answer to the question of Kronheimer as follows:

Theorem 1.3. *The product 4-manifold $S^1 \times M_K$ admits a symplectic structure if and only if the knot K is always fibered.*

According to the recent paper [10] of S. Friedl and S. Vidussi, the product of S^1 with the 0-surgery of S^3 along the pretzel knot $P(5, -3, 5)$ does not admit a symplectic structure, which fits well with our result. We give the proof of Theorem 1.3 at the end of Section 4. As a corollary, as the pretzel knot $P(5, -3, 5)$ shows, the statement that the symmetrized Alexander polynomial of a knot K is monic and its genus $g(K)$ is equal to the degree of its symmetrized Alexander polynomial does not imply that $S^1 \times M_K$ admits a symplectic structure. In summary, when we set the statements **(A)**, **(B)**, and **(C)** as follows,

- (A)** $S^1 \times M_K$ admits a symplectic structure,
- (B)** The symmetrized Alexander polynomial of a knot K with genus ≥ 2 is monic and its knot genus $g(K)$ is equal to the degree of its symmetrized Alexander polynomial,
- (C)** The knot K is fibered,

we can establish the following table for the various implications:

Table I

Implication	True/False	Reason
$(\mathbf{A}) \rightarrow (\mathbf{B})$	True	Proved by Kronheimer and Fintushel-Stern
$(\mathbf{B}) \rightarrow (\mathbf{A})$	False	e.g., Pretzel knot $P(5, -3, 5)$
$(\mathbf{B}) \rightarrow (\mathbf{C})$	False	e.g., Pretzel knot $P(5, -3, 5)$
$(\mathbf{C}) \rightarrow (\mathbf{B})^\dagger$	True	Proposition 8.16 in [2] (Neuwirth)
$(\mathbf{A}) \rightarrow (\mathbf{C})$	True	Theorem 1.3
$(\mathbf{C}) \rightarrow (\mathbf{A})$	True	Proved by Thurston

† For this direction, we do not need the restriction on the genus of a knot.

Finally a few remarks are in order. During the preparation of our paper, two papers related to the Conjecture 1.1 have appeared. In their paper [21], Kutluhan and Taubes studied the Seiberg-Witten Floer homology of M , assuming that $S^1 \times M$ admits a symplectic form. As a consequence, by combining their results with Theorem 1 of Y. Ni in [26], they gave a different proof that M fibers over S^1 , in case that M has the first Betti number equal to 1 and the first Chern class of the canonical line bundle is not torsion. Friedl and Vidussi also posted a preprint [13] asserting the proof of Conjecture 1.1 modulo some technical step regarding the residually finite solvability of $\pi_1(M)$ which allegedly depends on a work under preparation by M. Aschenbrenner and S. Friedl. Among other things, the twisted Alexander polynomials, algebraic group theory, and Stallings' characterization ([29]) for the fibration of a 3-manifold over a circle play crucial roles in their proof.

We organize this paper as follows. In Section 2, we give some basic facts about open book decompositions for closed contact 3-manifolds, partial open book decompositions for compact contact 3-manifolds with convex boundary, and confoliations. Section 3 is one of the key sections for this paper. In that section, we show that every symplectic class on $S^1 \times M$ can always be represented by a symplectic form invariant under the naturally defined S^1 -action. Finally Section 4 is devoted to the proofs of the main Theorems 4.2 and 4.3.

2 Giroux correspondence and confoliations

The aim of this section is to review some basic facts about open book decompositions for contact 3-manifolds, partial open book decompositions for compact contact 3-manifolds with convex boundary, and confoliations.

First we briefly review the definition of an open book decomposition of a closed 3-manifold M , and its extension to compact contact 3-manifolds with convex boundary can be easily obtained with an obvious modification (see

the recent papers [18] and [6]). Let (F, h) be a pair consisting of an oriented surface F and a diffeomorphism $h : F \rightarrow F$ which is the identity on ∂F , and K be a link in M . An open book decomposition for M with binding K is the quotient space

$$((F \times [0, 1]) / \sim_h, (\partial F \times [0, 1]) / \sim_h))$$

which is homeomorphic to M . Here the equivalence relation \sim_h is given by

$$\begin{aligned} (x, 1) &\sim_h (h(x), 0) \text{ for } x \in F \text{ and} \\ (x, t) &\sim_h (x, t') \text{ for } x \in \partial F \text{ and all } t, t' \in [0, 1]. \end{aligned}$$

We will call $F \times \{t\}$ for $t \in [0, 1]$ a *page* of the open book decomposition. Two open book decomposition is *equivalent* if there is an ambient isotopy between them taking binding to binding and pages to pages. We can obtain a new open book decomposition (F, h') from (F, h) by a *positive* (resp. *negative*) *stabilization*. Namely, F' is obtained from F by attaching a 1-handle B along ∂F and h' is obtained by extending h by the identity map on the 1-handle B and taking the composition $R_\gamma \circ h$ (resp. $R_\gamma^{-1} \circ h$) with the right-handed Dehn twist R_γ along a simple closed curve γ in F' dual to the core of the 1-handle B .

It is known that every closed 3-manifold has an open book decomposition, but it is not unique. A contact structure τ is said to be *supported* (or *adapted*) by the open book decomposition (F, h, K) if there is a contact 1-form λ satisfying the following properties:

- λ induces a symplectic form $d\lambda$ on each fiber F .
- K is transverse to τ , and the orientation on K given by λ is the same as the boundary orientation induced from F coming from the symplectic structure.

Thurston and Winkelnkemper showed in [33] that any open book decomposition (F, h, K) supports a contact structure. The contact planes constructed by them can be made arbitrary close to the tangent planes of the pages away from the binding. Recently E. Giroux showed in [15] that the converse also holds. To be more precise, the following theorem holds:

Theorem 2.1. *Every contact structures τ on a closed 3-manifold M is supported by some open book decomposition (F, h, K) . Moreover, two open book decompositions (F, h, K) and (F', h', K') which support the same contact structure (M, τ) become equivalent after applying a sequence of positive stabilizations to each.*

In our situation, we do not use the full version of this theorem. Rather we will need the result of Giroux to choose a coordinate chart on $S^1 \times M$ with which we can easily calculate the Lie derivative of the symplectic form for our purposes (e.g., see Lemma 3.4 for more details). Even if the above Theorem 2.1 is stated for closed contact 3-manifold, the construction of Giroux shows that the same result holds for contact 3-manifolds with contact boundary, as the papers [18] and [6] of Honda-Kazez-Matić and Egtü-Ozbagci show.

For the sake of reader's convenience, we briefly review the relative Giroux correspondence for compact contact 3-manifolds with convex boundary, although in the present paper we do not need the full strength of this correspondence. For more details, see [18] and [6], and most of what is presented here can be found in those two papers.

We first begin with the abstract version of a *partial open book decomposition* which is a triple (S, P, h) satisfying the following three properties:

- S is a compact oriented connected surface with non-empty boundary ∂S ,
- $P = P_1 \cup P_2 \cup \dots \cup P_r$ where P_1, P_2, \dots, P_r are 1-handles is a proper, but not necessarily connected, subsurface of S such that S is obtained from the closure of $S \setminus P$ by attaching 1-handles P_1, P_2, \dots, P_r successively,
- $h : P \rightarrow S$ is an embedding such that $h|_{\partial P \cup \partial S} = \text{identity}$.

Given a partial open book decomposition (S, P, h) , we can construct a compact 3-manifold with boundary as follows. Let $H = (S \times [-1, 0]) / \sim$, where $(x, t) \sim (x, t')$ for all $x \in \partial S$ and $t, t' \in [-1, 0]$, which is a solid handlebody with $S \times \{0\} \cup -S \times \{-1\}$ as the boundary under the obvious relation $(x, 0) \sim (x, -1)$ for all $x \in \partial S$. We also let $N = (P \times [0, 1]) / \sim$, where $(x, t) \sim (x, t')$ for all $x \in \partial P \cap \partial S$ and $t, t' \in [0, 1]$. Again each component of N is a solid handlebody whose boundary can be described by the connected arcs of the closure of $\partial P \setminus \partial S$. In other words, let c_1, c_2, \dots, c_n denote such connected arcs. Then each disk $D_i = (c_i \times [0, 1]) / \sim$ is contained in the boundary of N . Thus the boundary of N consists of the union of the disjoint disks D_i 's and the surface $P \times \{1\} \cup -P \times \{0\}$ with the relation $(x, 0) \sim (x, 1)$ for all $x \in \partial P \cap \partial S$.

Now let $M = N \cup H$ with the identification of $P \times \{0\} \subset \partial N$ (resp. $P \times \{1\} \subset \partial N$) with $P \times \{0\} \subset \partial H$ (resp. $h(P) \times \{-1\} \subset \partial H$). Then M is an oriented compact 3-manifold with oriented boundary

$$(2.1) \quad \partial M = (S \setminus P) \times \{0\} \cup -(S \setminus h(P)) \times \{-1\} \cup (\overline{\partial P \setminus \partial S}) \times [0, 1]$$

with the suitable identifications. If a compact 3-manifold M with boundary is obtained from the abstract partial open book decomposition (S, P, h) as above, then the triple (S, P, h) is called a *partial open book decomposition* of M . The notions such as compatibility of a contact structure with respect to a partial open book decomposition, the isomorphism class of two partial open book decompositions, and the definition of a positive stabilization of a partial open book decomposition can also be interpreted suitably for this relative version (e.g., see Definitions 1.10, 1.11, and 1.13 in [6]).

Recall that a closed oriented embedded surface Σ in a contact manifold (M, ξ) is called *convex* if there is vector field transverse to Σ which preserves the contact structure ξ . A generic surface Σ inside a contact manifold can be made convex (cf. [14] and Section 2.2 of [17]). So the assumption that the boundary be convex can be imposed without loss of generality.

In [18], Honda-Kazez-Matić associated the isomorphism classes of compact contact 3-manifolds with convex boundary to the isomorphism classes of partial open book decompositions modulo positive stabilizations. Conversely, in [6] Etgü and Ozbagci constructed its inverse by describing a compact contact 3-manifold with convex boundary compatible with a given partial open book decomposition. As in the proof of Proposition 1.9 in [6], such a construction is essentially given by the explicit construction of Thurston and Winkelnkemper. Hence the property, as well as others, that for closed contact 3-manifolds the contact planes constructed by them can be made arbitrary close to the tangent planes of the pages away from the binding can also be used for compact 3-manifolds with convex boundary.

Now we can state a relative version of Giroux correspondence as follows, which is a relative version of Giroux correspondence for closed contact 3-manifolds (see Theorem 0.1 in [6]).

Theorem 2.2. *There is a one-to-one correspondence between isomorphism classes of partial open book decompositions modulo positive stabilization and the isomorphism classes of compact contact 3-manifolds with convex boundary.*

In what follows, we will also need to use the result of Eliashberg and Thurston (Theorem 2.4.1 in [4]). A plane field $\eta = \ker \theta$ on an oriented 3-manifold is called *positive* (resp. *negative*) *confoliation* if $\theta \wedge d\theta \geq 0$ (resp. $\theta \wedge d\theta \leq 0$). Let us denote by ζ the product foliation of the manifold $S^2 \times S^1$ by the spheres $S^2 \times \{z\}$ for $z \in S^1$.

Theorem 2.3 (Eliashberg-Thurston). *Suppose that a C^2 -confoliation η on an oriented 3-manifold is different from the foliation ζ on $S^2 \times S^1$. Then η*

can be C^0 -approximated by contact structure. When η is a foliation it can be C^0 -approximated both by positive and negative contact structure.

3 Existence of S^1 -invariant symplectic structures

Recall that there exists an obvious circle action on the 4-manifold $S^1 \times M$ obtained by rotating the first factor of $S^1 \times M$. The aim of this section is to show that every symplectic class on $S^1 \times M$ can be represented by a symplectic form which is invariant under the obvious action of S^1 .

In what follows, we assume that $S^1 \times M$ admits a symplectic structure ω . If M is $S^2 \times S^1$, then clearly M fibers over S^1 . So from now on we also assume that M is not $S^1 \times S^2$, unless stated otherwise. Let X be the fundamental vector field associated to the action of S^1 , and let $\alpha = \iota_X \omega$. Since ω is a symplectic 2-form, α is clearly a nowhere vanishing 1-form on $S^1 \times M$. Now choose an arbitrary point t in S^1 . Let j_t denote the inclusion from M into $S^1 \times M$ given by $x \mapsto (t, x)$. Then we obtain a nowhere vanishing 1-form β_t by the pull-back of the 1-form α restricted to $\{t\} \times M$ via the inclusion j_t . In other words, $\beta_t = j_t^*(\alpha|_{\{t\} \times M})$. Then we have the following proposition.

Proposition 3.1. *The differential 3-form $\beta_t \wedge d_M \beta_t$ should vanish identically for all $t \in S^1$.*

Proof. We divide the proof into the following three cases:

(Case 1) First of all, assume that $\beta_t \wedge d_M \beta_t$ is non-zero for all t . Thus the 2-plane field $\xi_t = \ker \beta_t$ is a family of contact structures on M . It is obvious that $d_M \beta_t$ is a nowhere vanishing 2-form on M . Moreover, the following holds.

Lemma 3.2. *Under our assumption, the Lie derivative $\mathcal{L}_X \omega$ is nowhere vanishing on $S^1 \times M$.*

Proof. The proof follows from the Cartan's formula. Indeed, it suffices to note that

$$(3.1) \quad \begin{aligned} 0 \neq d_M \beta_t &= d_M j_t^*(\iota_X \omega) = j_t^* d \iota_X \omega = j_t^*(d \iota_X \omega + \iota_X d \omega) \\ &= j_t^*(\mathcal{L}_X \omega). \end{aligned}$$

This completes the proof of Lemma 3.2. □

Next we can show the following

Lemma 3.3. *The symplectic 2-form ω restricted to the contact structure $\xi_t = \ker \beta_t$ along $\{t\} \times M$ is non-zero.*

Proof. To see it, first note that there exists a Reeb vector field Z_t on M , depending on the parameter t , such that

$$(3.2) \quad 1 = \beta_t(Z_t) = j_t^*(\iota_X \omega)(Z_t) = \omega(X, (j_t)_*(Z_t)).$$

Since, along each point (t, x) in $S^1 \times M$, the vector space spanned by X and $(j_t)_*(Z_t)$ is transversal to the contact plane ξ_t and the equation (3.2) is satisfied, we conclude that the restriction $\omega|_{\xi_t}$ of the symplectic form ω on $S^1 \times M$ is non-zero. \square

Now, we apply the result of Giroux in [15] concerning the open book decomposition of a contact 3-manifold (Theorem 2.2). In our situation, we can choose a family of open book decompositions along a connected binding B_t associated to the contact structure β_t on M , so that the parameter s_t for the base manifold S^1 for the fibration associated to the open book decomposition is given by the Reeb vector field Z_t . We recall that by the way of the construction of the open book decomposition the contact plane ξ_t can be made arbitrarily close to the pages F_t . Thus it follows from Lemma 3.3 that, along $\{t\} \times M$, ω restricted to the pages F_t of the open book decomposition is also non-zero. That is, we see that, along $\{t\} \times M$, ω restricted to the pages F_t of the open book decomposition is a volume form away from the binding B_t . Recall also that $d_M \beta_t$ is a volume form on the pages F_t away from the binding B_t by the construction of the open book decomposition.

Let M_{B_t} be the result of 0-surgery along B_t . Then we have a fibration

$$\pi : S^1 \times M_{B_t} \rightarrow S^1 \times S^1,$$

and t and s_t will denote the first and second angular coordinates on the base manifold $S^1 \times S^1$ of the fibration π , respectively. Let $N(B_t)$ denote a tubular neighborhood of the binding B_t . Since, along $\{t\} \times M$, $d_M \beta_t$ and ω are both nowhere vanishing 2-forms restricted to the contact plane ξ_t , we can choose a smooth function f defined over $S^1 \times (M_B \setminus N(B))$ satisfying the following two properties:

- f is nowhere vanishing over $S^1 \times (M_{B_t} \setminus N(B_t))$ and
- $f \cdot d_M \beta_t$ coincides with ω , when restricted to the contact plane ξ_t .

Then the following lemma holds.

Lemma 3.4. *On the manifold $S^1 \times (M_{B_t} \setminus N(B_t))$ which can be identified with $S^1 \times (M \setminus N(B_t))$, the symplectic 2-form ω can be written locally as the form*

$$(3.3) \quad \omega = \pi^*(dt \wedge ds_t) + f(t, x)d_M\beta_t + ds_t \wedge \delta,$$

where δ is a 1-form on $S^1 \times (M \setminus N(B_t))$ and x denotes a local coordinate on M .

Proof. To see it, notice first that $\omega(Z_t, W_t)$ may be non-zero for the Reeb vector field Z_t and $W_t \in \xi_t$, while $\omega(X, W_t)$ should be zero for $W_t \in \xi_t$. Thus in local coordinates the symplectic form ω should have only the terms involving $\pi^*(dt \wedge ds_t)$, $d_M\beta_t$ and $ds_t \wedge \delta$. Due to the equation (3.2), the coefficient of $\pi^*(dt \wedge ds_t)$ should be 1, as stated. Thus we are done. \square

Finally, over $S^1 \times (M_{B_t} \setminus N(B_t))$ we compute the Lie derivative $\mathcal{L}_X\omega$ explicitly. To do so, note that we have

$$(3.4) \quad \begin{aligned} \mathcal{L}_X\omega &= d_{\iota_X}(\pi^*(dt \wedge ds_t) + f(t, x)d_M\beta_t + ds_t \wedge \delta) \\ &= d(-\iota_X\delta) \wedge ds_t. \end{aligned}$$

However, since we have

$$1 = \omega(X, (j_t)_*(Z_t)) = 1 - \iota_X\delta$$

by the equations (3.2) and (3.3), $\iota_X\delta$ should be zero. Thus it follows from (3.4) that $\mathcal{L}_X\omega = 0$. This clearly contradicts to Lemma 3.2. That is, this case does not occur.

(Case 2) We next assume that, for some $t = t_0$ in S^1 , $\beta_t \wedge d_M\beta_t$ is non-zero for all $x \in M$. By the continuity of smooth differential forms, $\beta_t \wedge d_M\beta_t$ should be non-zero for all t in some sufficiently small open interval I of t_0 and all $x \in M$.

Then apply the arguments in (Case 1) to the manifold $I \times M$ instead of $S^1 \times M$. Then we can also derive a contradiction in this case. In more detail, the manifold $I \times M$ admits a symplectic structure, denoted ω , by the restriction of the symplectic form ω on $S^1 \times M$ to $I \times M$. There still exists the fundamental vector field X on $I \times M$ associated to the natural action of S^1 on $S^1 \times M$, since the interval is regarded as an open submanifold of S^1 . However, clearly there is no S^1 -action on I . Using this fundamental vector field X on $I \times M$ and the fact that $\beta_t \wedge d_M\beta_t$ is non-zero on $I \times M$, as in Lemma 3.2 we can show that the Lie derivative $\mathcal{L}_X\omega$ is nowhere vanishing on $I \times M$. Furthermore, one can check that other arguments as well as Lemmas

3.3 and 3.4 go through without any modification. So, we can conclude that this case does not occur, either.

(Case 3) In this case we assume that, for some $t = t_0$ in S^1 and some $x_0 \in M$, $\beta_t \wedge d_M \beta_t$ is non-zero. Once again it follows from the continuity of smooth differential forms that $\beta_t \wedge d_M \beta_t$ should be non-zero for all t in some sufficiently small open interval I of t_0 and some $x_0 \in M$.

In order to apply the arguments of the previous cases, we need to take the contact part

$$V(\beta_t) = \{x \in M \mid \beta_t \wedge d_M \beta_t \neq 0 \text{ for } t \in I\}.$$

For simplicity, for each $t \in I$ we shall denote by W_t the closure of the connected component of the contact part $V(\beta_t)$ which contains x_0 . Then for each $t \in I$, W_t is a compact contact submanifold of M of codimension 0 with (possibly empty) boundary, since $\beta_t \wedge d_M \beta_t$ is a nowhere vanishing 3-form on $V(\beta_t)$ and so is a volume form there. Fortunately, in the present paper we do not need to know the precise information of W_t , unlike to the case in the paper [19].

Assume now that the boundary is non-empty. (Otherwise, we are reduced to (Case 2).) Then, as already mentioned in Section 2, we may assume without loss of generality that the boundary is convex. Thus for each $t \in I$ we obtain a compact contact 3-manifold W_t with convex boundary. It is also true that as in (Case 2) above there still exists the fundamental vector field X on $\cup_{t \in I} \{t\} \times W_t \subset S^1 \times M$ associated to the natural action of S^1 on $S^1 \times M$, since the interval I is again regarded as an open submanifold of S^1 . But this case is slightly different from (Case 2) in that W_t is a compact contact 3-manifold with convex boundary. So we need to use the relative Giroux correspondence (Theorem 2.2) instead of the Giroux correspondence (Theorem 2.1). In other words, for each $t \in I$ apply Theorem 2.2 to W_t in order to obtain its partial open book decomposition (S_t, P_t, h_t) . Thus W_t can now be described as the gluing of two handlebodies H_t and N_t by the map h_t whose boundary is given as in (2.1). However, since all the arguments in (Case 1) and (Case 2) are essentially local, those arguments applied to the compact contact 3-manifold W_t with convex boundary and symplectic 4-manifold $\cup_{t \in I} \{t\} \times W_t$ equipped with the symplectic form induced from $S^1 \times M$ will again go through without any modification. This in turn gives rise to a contradiction for this case, which means that this case does not occur, either.

This completes the proof of Proposition 3.1. \square

With this understood, the following theorem will play a crucial role in the proof of Theorem 4.2.

Theorem 3.5. *The symplectic class $[\omega]$ on $S^1 \times M$ can be represented by a symplectic form which is invariant under the obvious action of S^1 .*

Proof. We prove this theorem by contradiction. That is, suppose that the cohomology class $[\omega]$ cannot be represented by any S^1 -invariant symplectic form ω under the obvious S^1 -action. Then we would have

$$(3.5) \quad \mathcal{L}_X \omega \neq 0.$$

Note also that by Proposition 3.1 the differential 3-form $\beta_t \wedge d_M \beta_t$ vanishes identically for all $t \in S^1$. Then there are two possibilities we have to consider: $d_M \beta_t$ vanishes identically on $\{t\} \times M$ for all $t \in S^1$ or not.

So, suppose first that $d_M \beta_t$ vanishes identically on M as well. Then it can be shown that the Lie derivative $\mathcal{L}_X \omega$ vanishes identically. To see it, notice that it follows from the identity (3.1) that we have $j_t^*(\mathcal{L}_X \omega) = 0$. Thus we have $\mathcal{L}_X \omega(Z_t, W_t) = 0$ for any vector fields Z_t and W_t on $\{t\} \times M$ for each $t \in S^1$. Moreover, since ω is a symplectic form on $S^1 \times M$, we can choose a Darboux chart in a neighborhood of a point (t, x) whose coordinate vectors are given by non-zero vector fields $X_0 = X$ and X_i ($i = 1, 2, 3$). With this coordinate chart, we have

$$\mathcal{L}_X \omega(X, X_i) = d\iota_X \omega(X, X_i) = X(\iota_X \omega(X_i)) = X(1) = 0,$$

which implies that $\mathcal{L}_X \omega(X, Y_t) = 0$ for all vector field Y_t of $\{t\} \times M$. Therefore, we can conclude that in this case $\mathcal{L}_X \omega$ vanishes identically. But this clearly contradicts to the assumption (3.5).

On the other hand, if $d_M \beta_t$ does not vanish on $\{t\} \times M$, we need to use the result of Eliashberg and Thurston about perturbing a confoliation into a contact structure. If the 3-manifold M is $S^2 \times S^1$, clearly M fibers over S^1 , as mentioned earlier. Thus we may assume that our foliation ξ_t is different from the foliation ζ on $S^2 \times S^1$. Now if we apply Theorem 2.3 to ξ_t then we have a contact structure $\tilde{\xi}_t = \ker \tilde{\beta}_t$ which is a C^0 -approximation to ξ_t . Since $\tilde{\xi}_t$ is a C^0 -approximation of ξ_t , the symplectic 2-form ω can also be C^0 -approximated by a symplectic 2-form $\tilde{\omega}$ on $S^1 \times M$ so that $\tilde{\beta}_t = j_t^*(\iota_X \tilde{\omega}|_{\{t\} \times M})$. So we are essentially led to the (Case 1), (Case 2), or (Case 3) of Proposition 3.1, which has already shown not to occur.

Therefore, for either case we have derived a contradiction under our assumption (3.5). This completes the proof of Theorem 3.5. \square

Remark 3.6. Note that Theorem 3.5 does not imply that any arbitrary symplectic form ω on $S^1 \times M$ is always S^1 -invariant under the S^1 -action on the first factor of $S^1 \times M$. This can be easily seen by taking M to be the 3-dimensional torus T^3 . That is, if the theorem implies that any symplectic form ω on $S^1 \times T^3$ is always S^1 -invariant under the obvious S^1 -action, the symplectic form on $S^1 \times T^3 = T^4$ should also be invariant under the obvious S^1 -action of the last three S^1 -factors of T^4 . So we can conclude that every symplectic form on T^4 should be invariant under the componentwise T^4 -action on T^4 . But obviously this is not the case for T^4 .

4 Proofs of Theorems 4.2 and 4.3

In this section we present the proofs of main theorems of the present paper. Once we have established the existence of an S^1 -invariant symplectic structure on $S^1 \times M$, it is a fairly standard procedure to complete the proof of Conjecture 1.1. For the sake of reader's convenience, we give its proof relatively in detail.

To do so, we begin with the following well-known lemma which says that when the cohomology class $[\iota_X \omega]$ is not integral and non-zero, by some suitable perturbation we can always make it integral.

Lemma 4.1. *Let ω be an S^1 -invariant symplectic form on a closed oriented 4-manifold N such that $[\iota_X \omega]$ is non-zero. Then N admits an S^1 -invariant symplectic form $\hat{\omega}$ such that $[\iota_X \hat{\omega}]$ is non-zero and integral.*

Proof. If N admits an S^1 -invariant symplectic form ω' such that $[\omega']$ is rational, then we can easily obtain an S^1 -invariant symplectic form $\hat{\omega}$ such that $[\hat{\omega}]$ is integral by multiplying some suitable integer to ω' . Note also that the class $[\iota_X \omega']$ is rational if the class $[\omega']$ is.

So assume now that the class $[\omega]$ is not rational. It is clear that there exists an arbitrary small closed 2-form η such that $\omega + \eta$ represent a rational cohomology class. Let $\hat{\eta}$ be the average of η over the S^1 -action. Since S^1 is connected, for $\nu \in S^1$ $\nu^* \eta$ is a closed 2-form representing the same cohomology class as η . Thus $\omega + \hat{\eta}$ and $\omega + \eta$ have the same rational cohomology class. Note also that $\omega' = \omega + \hat{\eta}$ is symplectic, provided that η is sufficiently small. By the openness of symplectic condition again, we can further choose ω' in such a way that the class $[\iota_X \omega']$ is non-zero. This completes the proof. \square

Finally we are ready to prove the main theorems.

Theorem 4.2. *Let M be a closed oriented 3-manifold such that $S^1 \times M$ admits a symplectic structure ω . Then M fibers over S^1 .*

Proof. By Theorem 3.5, we may assume that the symplectic structure ω is S^1 -invariant. Further, we may also assume that the class $[\iota_X\omega]$ on $S^1 \times M$ is integral by Lemma 4.1.

In order to prove the theorem, we first consider the case where the class $[\iota_X\omega]$ is zero. In this case, there exists a function, called the moment map $\mu : S^1 \times M \rightarrow \mathbf{R}$ such that $\iota_X\omega = d\mu$. Thus the S^1 -action is Hamiltonian. But it is clear that the S^1 -action on $S^1 \times M$ does not have any fixed points that are critical points of μ . This gives rise to a contradiction to the fact that any Hamiltonian function on a closed symplectic manifold should have at least two critical points (e.g., extremal points). Therefore we can conclude that the class $[\iota_X\omega]$ is actually non-zero. Under this condition, McDuff proved in [23] that by using an argument of D. Tischler in [32], there exists a generalized moment map $\mu : S^1 \times M \rightarrow S^1$ satisfying $\iota_X\omega = \mu^*(dt)$. Thus by restricting the map μ to $\{\text{a point}\} \times M$, we easily obtain a fibration of M over S^1 . This completes the proof of Theorem 4.2. \square

Now we close this section with a proof of Theorem 1.3 as follows.

Theorem 4.3. *If $S^1 \times M_K$ admits a symplectic structure, then K is always a fibered knot.*

Proof. Suppose that $S^1 \times M_K$ admits a symplectic structure. Then it follows from Theorem 4.2 that the 3-manifold M_K is a fibration of S^1 . Moreover, by the construction in the proof of Theorem 4.2, we have a closed 1-form $\iota_X\omega$ whose class is integral and is pointwise non-zero. Now let $j : S^3 - N(K) \rightarrow M_K$ be the natural inclusion, where $N(K)$ is a tubular neighborhood of K . Now observe that by the pullback we have a closed 1-form $j^*(\iota_X\omega)$ on $S^3 - N(K)$ whose class is still integral and is non-zero pointwise on $S^3 - N(K)$. Then $j^*(\iota_X\omega)$ defines a measured foliation \mathcal{F} of $S^3 - N(K)$ transverse to the boundary $\partial(S^3 - N(K))$ with $T\mathcal{F} = \ker j^*(\iota_X\omega)$. Since $j^*(\iota_X\omega)$ is integral, we can also write $j^*(\iota_X\omega) = d\pi$ for a fibration $\pi : S^3 - N(K) \rightarrow S^1$ whose fibers are the leaves of \mathcal{F} (e.g., see p.2 of [25] for more details). Thus the knot K is indeed fibered. This completes the proof. \square

References

- [1] S. Baldridge, *Seiberg-Witten invariants of 4-manifolds with free circle actions*, Commun. Contemp. Math. **3** (2001), 341–353.
- [2] G. Burde and H. Zieschang, *Knots*, de Gruyter Stud. Math. **5**, 1985.

- [3] W. Chen and R. Matveyev, *Symplectic Lefschetz fibrations on $S^1 \times M^3$* , *Geom. Topo.* **4** (2000), 517–535.
- [4] Y. Eliashberg and W. Thurston, *Confoliations*, University Lecture Series, American Mathematical Society, 1997.
- [5] T. Etgü, *Lefschetz fibrations, complex structures and Seifert fibrations on $S^1 \times M^3$* , *Alg. Geo. Topo.* **1** (2001), 469–489.
- [6] T. Etgü and B. Ozbagci, *Relative Giroux correspondence*, preprint (2008); arXiv:0802.0810v1.
- [7] J. Etnyre, *Lectures on open book decompositions and contact structures*, preprint (2004); math.SG/0409402.
- [8] M. Fernández, A. Gray, and J. Morgan, *Compact symplectic manifolds with free circle actions, and Massey products*, *Michigan Math. J.* **38** (1991), 271–283.
- [9] R. Fintushel and R. Stern, *Knots, links, and four-manifolds*, *Invent. Math.* **134** (1998), 363–400.
- [10] S. Friedl and S. Vidussi, *Twisted Alexander polynomials and symplectic structures*, *American J. Math.* **130** (2008), 455–484.
- [11] S. Friedl and S. Vidussi, *Symplectic $S^1 \times N^3$, subgroup separability, and vanishing Thurston norm*, *J. American Math. Soc.* **21** (2008), 597–610.
- [12] S. Friedl and S. Vidussi, *Symplectic 4-manifolds with a free circle action*, preprint (2008); arXiv:0801.1513v1.
- [13] S. Friedl and S. Vidussi, *Twisted Alexander polynomials detect fibred 3-manifolds*, preprint (2008); arXiv:0805.1234v1.
- [14] E. Giroux, *Convexité en topologie de contact*, *Comm. Math. Helv.* **66** (1991), 637–677.
- [15] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, *Proceedings of the International Congress of Mathematics, Vol. II* (Beijing, 2002), 405–414.
- [16] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, *Grad. Stud. Math.* **20**, Amer. Math. Soc., 1999.

- [17] K. Honda, *On the classification of tight contact structures I*, Geom. Topo. **4** (2000), 309–368.
- [18] K. Honda, W. Kazez, and G. Matić, *The contact invariant in the sutured Floer homology*, preprint (2007); arXiv:math.GT/0705.2828v2.
- [19] M. Hutchings and C. Taubes, *The Weinstein conjecture for stable Hamiltonian structures*, preprint (2008); arXiv:0809.0140v1.
- [20] P. Kronheimer, *Minimal genus in $S^1 \times M^3$* , Invent. Math. **135** (1999), 45–61.
- [21] C. Kutluhan and C. Taubes, *Seiberg-Witten Floer homology and symplectic forms on $S^1 \times M^3$* , preprint (2008); arXiv:0804.1371v3.
- [22] J. McCarthy, *On the asphericity of a symplectic $S^1 \times M^3$* , Proc. Amer. Math. Soc. **129** (2001), 257–264.
- [23] D. McDuff, *The moment map for the circle actions on symplectic manifolds*, Jour. Geom. Phys. **5** (1988), 149–160.
- [24] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford Science Publ., 1998.
- [25] C. McMullen and C. Taubes, *4-manifolds with inequivalent symplectic forms and 3-manifolds with inequivalent fibrations*, Math. Res. Lett. **6** (1999), 681–696.
- [26] Y. Ni, *Addendum to: “Knots, sutures, and excision”*, preprint (2008); arXiv:0808.1327v1.
- [27] K. Ono, *private communication*.
- [28] D. Rolfsen, *Knots and Links*, Math. Lect. Notes **7**, 1990.
- [29] J. Stallings, *On fibering certain 3-manifolds*, Topology of 3-manifolds and related topics, Prentice-Hall, (1962), 95–100.
- [30] C. Taubes, *The geometry of the Seiberg-Witten invariants*, Doc. Math. Extra Vol. II (1998), 439–504.
- [31] W. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), 467–468.
- [32] D. Tischler, *On fibering certain foliated manifolds over S^1* , Topology **9** (1970), 153–154.

- [33] W. Thurston and H. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347.